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Maximum growth rate of magnetoatmospheric instabilities: II. A Hilbert space approach

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Abstract. An upper bound is derived on the growth rates of instabilities occurring in media with restoring forces due to compressibility, buoyancy and magnetic fields (a magnetoatmosphere). The magnetic field considered is horizontal but arbitrarily varying in the vertical direction. The formalism used is based on the concept of an energy inner product in Hilbert space.

In this paper we examine the equations describing motion in a class of media called magnetoatmospheres in which the effects of buoyancy, compressibility and magnetic fields are of importance in sustaining wave motion. This is a rigorous generalisation of previous normal-mode work (Adam 1977) using an approach of Rosencrans (1969) on non-magnetic media. For ease of comparison we use a similar notation.

The linearised equations of momentum, continuity, state and induction are

$$\rho_0\, \partial u_1/\partial t = -\partial p_1/\partial x_1 + B_0' b_3/4\pi, \tag{1}$$

$$\rho_0 \frac{\partial u_2}{\partial t} = -\frac{\partial p_1}{\partial x_2} - \frac{B_0}{4\pi}\left(\frac{\partial b_1}{\partial x_2} - \frac{\partial b_2}{\partial x_1}\right), \tag{2}$$

$$\rho_0 \frac{\partial u_3}{\partial t} = -\frac{\partial p_1}{\partial x_3} - \rho_1 g - \frac{B_0}{4\pi}\left(\frac{\partial b_1}{\partial x_3} - \frac{\partial b_3}{\partial x_1}\right) - \frac{B_0'}{4\pi} b_1, \tag{3}$$

$$\frac{\partial \rho_1}{\partial t} = -\rho_0\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) - \rho_0' u_3\,, \tag{4}$$

$$\partial s_1/\partial t = -s_0' u_3, \tag{5}$$

$$p_1 = p_0 s_1/c_v + c_0^2 \rho_1, \tag{6}$$

$$\partial b_1/\partial t = -B_0(\partial u_2/\partial x_2 + \partial u_3/\partial x_3) - B_0' u_3, \tag{7}$$

$$\partial b_2/\partial t = B_0\, \partial u_2/\partial x_1, \tag{8}$$

$$\partial b_3/\partial t = B_0\, \partial u_3/\partial x_1. \tag{9}$$

In these equations (u_1, u_2, u_3) is the fluid velocity, (b_1, b_2, b_3) is the magnetic field perturbation, ρ, p, s refer to density, pressure and entropy respectively. The perturbations of these last three quantities from the equilibrium state (denoted by subscript zero) are denoted by subscript one. The equilibrium states of the magnetic field, density, pressure and entropy are all functions of the upward vertical ordinate x_3. The magnetic field taken here is $\boldsymbol{B}_0 = (B_0(x_3), 0, 0)$, gravitational acceleration (constant)

$\boldsymbol{g} = (0, 0, -g)$, $\gamma = c_p/c_v$ is the ratio of specific heats and $c_0^2 = \gamma p/p_0$ is the square of the local velocity of sound. It is also assumed that the medium is a perfect electrical conductor.

Primes refer to differentiation with respect to x_3.

Equilibrium quantities satisfy

$$p_0 = K \, e^{s_0/c_v} \rho_0^\gamma, \qquad K \text{ constant} \tag{10}$$

and

$$d/dx_3 \, (p_0 + B_0^2/8\pi) = -\rho_0 g, \tag{11}$$

with $u_i = 0$, $i = 1, 2, 3$. $s_0(x_3)$ is prescribed.

As stated by Rosencrans (1969), in the absence of any magnetic field, Schwarzschild's criterion states that the system is stable if $s_0'(x_3) \geq 0$ for all x_3, and unstable if $s_0'(x_3) < 0$ for at least some x_3 (s_0 assumed smooth and s_0' bounded). We wish to determine an upper bound on the growth rate of instabilities when the horizontal magnetic field is present.

Define

$$\omega(x) = \begin{cases} (g s_0'(x_3)/c_p)^{1/2} & \text{if } s_0'(x_3) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma(x) = \begin{cases} (-g s_0'(x_3)/c_p)^{1/2} & \text{if } s_0'(x_3) < 0 \\ 0 & \text{otherwise;} \end{cases}$$

when $s_0'(x_3) < 0$, $\omega(x)$ is just the Brunt–Vaisala frequency of oscillation of a fluid particle when vertically displaced from equilibrium.

Define also $\Omega = \omega + i\sigma$ (clearly $\Omega^2 = g s_0'/c_p$). Hence Schwarzschild's criterion can be stated as: the system is convectively stable and only if (for zero magnetic field)

$$\tilde{\sigma} = \sup_x \sigma(x) = 0 \qquad (B_0 = 0).$$

We now define new variables

$$v_i = (\rho_0 c_0)^{1/2} u_i, \qquad i = 1, 2, 3, \qquad v_4 = -(\rho_0 c_0)^{-1/2} p_1,$$

$$v_s = g(\rho_0 c_0)^{1/2} \Omega^{-1} s_1 c_p^{-1}, \qquad v_j = (c_0/4\pi)^{1/2} b_i, \qquad j = 6, 7, 8,$$

where the j's and i's correspond. Thus by eliminating ρ_1 in the above equations we obtain the system

$$c_0^{-1} \partial v_1/\partial t = \partial v_4/\partial x_1 + \tilde{\beta} v_8, \tag{12}$$

$$c_0^{-1} \frac{\partial v_2}{\partial t} = \frac{\partial v_4}{\partial x_2} - \beta\left(\frac{\partial v_6}{\partial x_2} - \frac{\partial v_7}{\partial x_1}\right), \tag{13}$$

$$c_0^{-1} \frac{\partial v_3}{\partial t} = \frac{\partial v_4}{\partial x_3} + \lambda_1 c_0^{-1} v_4 + \Omega c_0^{-1} v_5 - \beta\left(\frac{\partial v_6}{\partial x_3} - \frac{\partial v_8}{\partial x_1}\right) + \frac{\beta v_6 c_0'}{2 c_0} - \tilde{\beta} v_6, \tag{14}$$

$$c_0^{-1} \frac{\partial v_4}{\partial t} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} - \lambda_2 c_0^{-1} v_3, \tag{15}$$

$$c_0^{-1} \partial v_5/\partial t = -\Omega c_0^{-1} v_3, \tag{16}$$

$$c_0^{-1}\frac{\partial v_6}{\partial t}=-\beta\left(\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)+\beta v_3\frac{(\rho_0 c_0)'}{\rho_0 c_0}-\tilde{\beta}v_3, \tag{17}$$

$$c_0^{-1}\partial v_7/\partial t=\beta\,\partial v_2/\partial x_1, \tag{18}$$

$$c_0^{-1}\partial v_8/\partial t=\beta\,\partial v_3/\partial x_1. \tag{19}$$

In these equations $\beta(x_3)=B_0/c_0(4\pi\rho_0)^{1/2}=a_0/c_0$ is the (local) ratio of Alfvén velocity $a_0(x_3)$ to sound velocity $c_0(x_3)$, $\tilde{\beta}(x_3)=B_0'/c_0(4\pi\rho_0)^{1/2}$ and

$$\lambda_1 c_0^{-1}=\frac{1}{(\rho_0 c_0)^{1/2}}\frac{\partial}{\partial x_3}(\rho_0 c_0)^{1/2}+\frac{g}{c_0^2},\qquad \lambda_2 c_0^{-1}=\frac{(\rho_0 c_0)'}{2\rho_0 c_0}-\frac{\rho_0'}{\rho_0}-\frac{s_0'}{c_p}.$$

It is readily seen that these two terms are equal, since for stable regions

$$\Omega^2=gs_0'/c_p=-g\rho_0'/\rho_0-g^2/c_0^2.$$

Hence we set $\lambda_1=\lambda_2=\lambda$.

Before proceeding further a point should be made concerning the normalisation procedure for obtaining the v_i. This is certainly somewhat artificial in the present context for a non-zero magnetic field, since there is an obvious asymmetry in a_0 and c_0 which could be simply amended by an appropriate new normalisation. However, the present form brings out *explicitly* the effect of the magnetic field on the maximum growth rate of instability compared with the zero field case; were we concerned with a suitable new definition of stability it would obviously be more appropriate to define ω and σ in terms of

$$[gs_0'/c_p+(g/\gamma p_0)(B_0^2/8\pi)'],$$

thus elucidating the stabilising (destabilising) effect of B_0' positive (negative). (This expression can be obtained very simply from equations (10) and (11).) This would of course give an upper bound on the growth rate with the details of the field configuration implicit in the new $\tilde{\sigma}$.

We now write the system (12)–(19) in matrix form as

$$c_0^{-1}\partial v/\partial t=(\mathrm{i}T+B)v, \tag{20}$$

where $v=(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8)$, T and $B=B_1+\mathrm{i}B_2+B_3+\mathrm{i}B_4$ are defined as follows:

$$T=\sum_{k=1}^{3}T^{(k)}\frac{\partial}{\mathrm{i}\partial x_k},$$

$$T_{lm}^{(k)}=\delta_{lk}\delta_{4m}+\delta_{4l}\delta_{mk}+\beta(-\delta_{2k}\delta_{2l}\delta_{6m}-\delta_{3k}\delta_{3l}\delta_{6m}+\delta_{1k}\delta_{2l}\delta_{7m}$$
$$+\delta_{1k}\delta_{3l}\delta_{8m}-\delta_{2k}\delta_{6k}\delta_{2m}-\delta_{3k}\delta_{6l}\delta_{3m}+\delta_{1k}\delta_{7l}\delta_{2m}+\delta_{1k}\delta_{8l}\delta_{3m})$$

where $k=1, 2, 3$, $l, m=1, 2, 3, 4, 5, 6, 7, 8$, and

$$B_{1(lm)}=\mathrm{i}\sigma c_0^{-1}(\delta_{l3}\delta_{5m}-\delta_{l5}\delta_{3m}),$$

$$B_{2(lm)}=\mathrm{i}c_0^{-1}[\lambda(\delta_{l4}\delta_{3m}-\delta_{l3}\delta_{4m})+\omega(\delta_{l5}\delta_{3m}-\delta_{l3}\delta_{5m})],$$

$$B_{3(lm)}=\tfrac{1}{2}[(a_2+a_3)(\delta_{l3}\delta_{6m}+\delta_{l6}\delta_{3m})+a_1(\delta_{l1}\delta_{8m}+\delta_{l8}\delta_{1m})],$$

$$B_{4(lm)}=-\tfrac{1}{2}\mathrm{i}[(a_2-a_3)(\delta_{l3}\delta_{6m}-\delta_{l6}\delta_{3m})+a_1(\delta_{l1}\delta_{3m}-\delta_{l8}\delta_{1m})],$$

with

$$a_1=\tilde{\beta},\qquad a_2=\beta c_0'/2c_0-\tilde{\beta},\qquad a_3=\beta(\rho_0 c_0)'/2\rho_0 c_0-\tilde{\beta}.$$

Thus $T^{(k)}$ is real and symmetric, as is B_3, while B_1, B_2 and B_4 are Hermitian. It may be noted that with the introduction of the terms in β arising from the presence of the magnetic field, equation (20) can no longer be expressed as a symmetric hyperbolic system in the sense of Friedrichs, with the present choice of variables, even if β is constant. We assume c_0 is bounded away from zero.

We define the energy inner product as

$$(u, v) = \int_{R^3} \bar{u}^*(x) c_0^{-1} v(x)\,\mathrm{d}x \tag{21}$$

where * denotes a matrix transpose, and '$\bar{\ }$' complex conjugation. This is motivated by the conservation of energy law which is satisfied by solutions of (20)

$$\frac{\partial}{\partial t}\left(\frac{u^* c_0^{-1} u}{2}\right) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{u^* T^{(k)} u}{2}\right) + \frac{1}{2} u^*(B + B^*)u = 0.$$

Thus the quadratic forms $u^* c_0^{-1} u$ and $u^* T^{(k)} u$ can be formally interpreted as energy density and flux of energy (energy per unit area per unit time) respectively.

Let H be the Hilbert space of measurable functions $u : R^3 \to C^8$ with finite energy, i.e. such that

$$\|u\|^2 = (u, u) < \infty.$$

Let the domain D of the formal operator $c_0(\mathrm{i}T + B)$ be the linear manifold in H of continuously differentiable functions of compact support.

Given a classical solution to (20), i.e. a function $v(x, t)$ continuously differentiable with respect to x and t, such that

$$v \to 0 \quad \text{as} \quad |x| \to \infty,$$

then the differential operator T is self-adjoint relative to the energy inner product, i.e. $(c_0 T v, v) = (v, c_0 T v)$. So

$$\begin{aligned}
(\mathrm{d}/\mathrm{d}t)\|v\|^2 &= (\partial/\partial t)\|v\|^2 \text{ in the linear approximation} \\
&= 2\,\mathrm{Re}(v, v_t) \\
&= 2(c_0 B_1 v, v) + 2(c_0 B_3 v, v) (\text{from the Hermitian nature of } B_1, B_2, B_4) \\
&= 2\int_{R^3} \overline{(B_1 v)^*} v\,\mathrm{d}x + 2\int_{R^3} \overline{(B_3 v)^*} v\,\mathrm{d}x \\
&\leq 2\int_{R^3} \sigma c_0^{-1} \bar{v}^* v\,\mathrm{d}x + 2\int_{R^3} c_0 \max_{\Lambda_j}(\Lambda_j) c_0^{-1} \bar{v}^* v\,\mathrm{d}x,
\end{aligned}$$

since the eigenvalues of B_1 are 0 and $\pm\sigma c_0^{-1}$, while we suppose for the moment that B_3 has eigenvalues Λ_j. Hence, if $\tilde{\Lambda} = \sup c_0 \max_{\Lambda_j}(\Lambda_j)$, (assuming this exists) then

$$(\mathrm{d}/\mathrm{d}t)\|v\|^2 \leq 2(\tilde{\sigma} + \tilde{\Lambda})\|v\|^2; \tag{22}$$

hence

$$\|v\| \leq \mathrm{e}^{(\tilde{\sigma}+\tilde{\Lambda})t}\|v_0\|, \tag{23}$$

where $v(x, 0) = v_0(x)$, $v_0 \in H$.

After some algebra it is found that the eigenvalue equation for B_3 is

$$\Lambda^4[\Lambda^2 - \tfrac{1}{4}(a_2 + a_3)^2](\Lambda^2 - \tfrac{1}{4}a_1^2) = 0$$

i.e.

$$\Lambda = 0,\ \pm\tfrac{1}{2}(a_2 + a_3),\ \pm\tfrac{1}{2}a_1.$$

Writing $a_i (i = 1, 2, 3)$ out in full, we see that

$$a_1 = \tilde{\beta} = B_0'/c_0(4\pi\rho_0)^{1/2} = \beta B_0'/B_0,$$

$$a_2 + a_3 = \frac{\beta}{2}\left[\frac{2c_0'}{c_0} + \frac{\rho_0'}{\rho_0}\right] - 2\tilde{\beta} = \beta\left[\frac{c_0'}{c_0} + \frac{\rho_0'}{2\rho_0} - \frac{B_0'}{B_0}\right] - \tilde{\beta} = -\beta'(x_3) - \tilde{\beta}(x_3).$$

$$\therefore\quad \tilde{\Lambda} = \sup_x c_0 \max[\pm\tilde{\beta}/2,\ \pm(\beta' - \tilde{\beta})/2]. \tag{24}$$

The appropriate eigenvalue chosen will depend on the details of the equilibrium distributions of the physical quantities. As two examples let us consider (i) $\beta =$ constant, (ii) $\tilde{\beta} = 0$.

Case (i) corresponds to a magnetic field $B_0(x_3)$ varying in such a way vertically that a_0/c_0 is constant—for example if c_0 is constant then $B_0(x_3) = B_0(0)\exp(-x_3/2H)$, where H is the constant density scale height. More generally, for $c_0 = c_0(x_3)$ the behaviour will be less simple. Then

$$\tilde{\Lambda} = \sup_x c_0\left|\frac{\beta B_0'}{2B_0}\right| = \sup_x \frac{a_0|\beta_0'|}{2B_0} = \sup_x \frac{|B_0'|}{2(4\pi\rho_0)^{1/2}}.$$

If the equilibrium configurations are such that $B_0' < 0$, this supremum is useful inasmuch as it implies that for this case (and indeed for $B_0' > 0$) $\|v\|$ is *not* bounded independently of t, even if $\tilde{\sigma} = 0$. It is easy to show that such a field configuration tends to destabilise an otherwise stable *equilibrium*, whereas if $B_0' > 0$ this tends to stabilise the pre-existing equilibrium, so (23) appears less useful. We must not confuse, however, the concepts of *onset* of instability and *growth rate* of instability, the upper limit of the latter being given by the energy inequality (23). Case (ii) corresponds to a constant magnetic field, and so

$$\tilde{\Lambda} = \sup_x \frac{c_0}{2}\left|\left(\frac{a_0}{c_0}\right)'\right| = \sup_x \frac{a_0}{2}\left|\frac{a_0'}{a_0} - \frac{c_0'}{c_0}\right| = \sup_x \frac{a_0}{2}\left|\frac{\rho_0'}{2\rho_0} + \frac{c_0'}{c_0}\right|.$$

Other examples could be chosen but this will not be done here. A point that should be noted is that the behaviour of (in particular) c_0, ρ_0 and a_0 in these examples must be such as to keep the suprema finite for large $|x|$, as, for example, will be the case if these quantities approach constant values (non-zero for ρ_0, c_0) as $|x| \to \infty$.

In conclusion therefore we have shown in a rigorous manner that *any* horizontal magnetic field (subject, with its derivatives, to appropriate boundedness and continuity conditions) affects the maximum growth rate of unstable motions in a magneto-atmosphere. This has been proved for the full system of partial differential equations; it contains the normal-mode result as a special case. In particular the specific maximum growth rates for (i) constant Alfvén velocity and (ii) constant magnetic field have been obtained. The latter confirms the result of Newcomb (1961) for a normal-mode type solution, namely that the presence of a horizontal magnetic field, while not affecting the stability of the equilibrium, certainly affects the growth rate of any instability.

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